Weighted pseudo-almost periodic functions on time scales with applications to cellular neural networks with discrete delays*

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Abstract

In this paper, we first propose a concept of weighted pseudo-almost periodic functions on time scales and study some basic properties of weighted pseudo-almost periodic functions on time scales. Then, we establish some results about the existence of weighted pseudo-almost periodic solutions to linear dynamic equations on time scales. Finally, as an application of our results, we study the existence and global exponential stability of weighted pseudo-almost periodic solutions for a class of cellular neural networks with discrete delays on time scales. The results of this paper are completely new.

Keywords: Weighted pesudo-almost periodic solutions; Global exponential stability; Neural networks; Exponential dichotomy; Time scales.
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1 Introduction

The concept of pseudo-almost periodicity, which is a natural generalization of the notion of almost periodicity, was introduced in the literature more than a decade ago by C. Zhang [1-3]. Since its introduction in the literature, the notion of pseudo-almost periodicity has generated several developments and extensions, see, e.g., [4, 5]. The concept of weighted pseudo-almost periodicity introduced in the literature in 2006 by Diagana [5]. The notion of weighted pseudo-almost periodicity has generated several developments since then, see for instance the references [6-8]. Among other things, it has been utilized to study the qualitative

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behavior to various differential and partial differential equations involving weighted pseudo-almost periodic coefficients, see, e.g., [9-11].

On the other hand, dynamic equations on time scales offer a new direction in the study of dynamic systems which involve differential equations and difference equations as special cases. Their origin is connected with Stefan Hilger's work [12, 13]. In 1988, Stefan Hilger introduced the definition of a Δ-derivative. The common derivative and the common forward difference are special cases of the Δ-derivative. In fact, the progressive field of dynamic equations on time scales contains links to and extends the classical theory of differential and difference equations. For instance, by choosing the time scale to be the set of real numbers, the general result yields a result for differential equations. In a similar way, by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. For these reasons, based on the concept of almost periodic time scales proposed in [14, 15], the concept of pseudo-almost periodic functions on almost periodic time scales was formally introduced by Li and Wang (2012) in [16]. Moreover, some first results were proven which concern the pseudo-almost periodic solution to dynamic equations on time scales. However, to the best of our knowledge, there is no concept of weighted pseudo-almost periodic functions on time scales yet, so up to now, there was no work on discussing weighted pseudo-almost periodic problems of dynamic equations on time scales before.

Also, it is well known that cellular neural networks have been extensively applied in areas of signal processing, image processing, pattern recognition, optimization and associative memories. Since all these applications closely relate to the dynamics, the dynamical behaviors of cellular neural networks have been widely investigated. There have been extensive results on the problem of the existence and stability of equilibrium points, periodic solutions and almost periodic solutions of cellular neural networks in the literature. We refer the reader to [17-27] and the references cited therein. However, to the best of our knowledge, few authors have studied the problems of pseudo-almost periodic and weighted pseudo-almost periodic solutions of neural networks [28, 29]. Moreover, it is known that the existence and stability of almost periodic solutions play a key role in characterizing the behavior of dynamical systems (see [30-36]) and pseudo-almost periodicity is a natural generalization of the notion of almost periodicity. Furthermore, because of the weights involved, the concept of weighted pseudo-almost periodicity is more general and richer than the concept of pseudo-almost periodicity.

Motivated by the above discussion, in this paper, we first introduce the concept of weighted pseudo-almost periodic functions on time scales and study some basic properties of weighted pseudo-almost periodic functions on time scales. Then, we establish some results about the existence of weighted pseudo-almost periodic solutions to linear dynamic equations on time scales. Finally, as an application of our results, we study the existence and global exponential stability of weighted pseudo-almost periodic solutions for the following cellular neural network with discrete delays on time scales

$$x_i^{\Delta}(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\gamma_{ij})) + I_i(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where $i=1,2,\ldots,n$ and $\mathbb{T}$ is an almost periodic time scale which will be defined in the next section, $x_i(t)$ correspond to the activations of the ith neurons at the time t, $c_i(t)$ are positive functions, they denote the rate with which the cell i reset their potential to the resting state when isolated from the other cells and inputs at time t, $a_{ij}(t)$ and $b_{ij}(t)$ are the connection weights at time t, γ_{ij} are nonnegative, which corresponds to the finite speed of the axonal signal transmission, $I_i(t)$ denote the external inputs at time t, f_i are the activation functions of signal transmission. For each interval J of $\mathbb{R}$, we denote by $J_{\mathbb{T}}=J\cap\mathbb{T}$.

The system (1.1) is supplemented with the initial values given by

$$x_i(s)=\varphi_i(s),\ i=1,2,\ldots,n,$$

where $\varphi_i(\cdot)$ denotes a real-value bounded right-dense continuous function defined on $[-\gamma,0]_{\mathbb{T}}$, and $\gamma_i=\max_{1\leq j\leq n}\{\gamma_{ij}\},\gamma=\max_{1\leq i\leq n}\{\gamma_i\}$.

Throughout this paper, we assume that the following conditions hold:

(H_1) $f_j\in C(\mathbb{R},\mathbb{R})$ and there exist positive constants α_j such that

$$|f_j(u)-f_j(v)|\leq\alpha_j|u-v|,\quad u,v\in\mathbb{R},\ j=1,2,\ldots,n;$$

(H_2) c_i,a_{ij},b_{ij} are almost periodic functions on $\mathbb{T}$, where $i,j=1,2,\ldots,n$;

(H_3) $\inf\limits_{t\in\mathbb{T}}c_i(t)>0,\ -c_i\in\mathcal{R}^+,\gamma_{ij}\in\Pi,\ i,j=1,2,\ldots,n$, where $\mathcal{R}^+$, Π will be defined in the next section;

(H_4) For fixed $u\in\mathbb{U}_\infty^{Inv}$. $I_i\,(i=1,2,\ldots,n)$ are weighted pseudo-almost periodic functions, where $\mathbb{U}_\infty^{Inv}$ will be defined in the next section.

Remark 1.1. *If* $\mathbb{T}=\mathbb{R}$, *then* (1.1) *reduces to the following form*

$$x_i'(t)=-c_i(t)x_i(t)+\sum_{j=1}^n a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^n b_{ij}(t)f_j(x_j(t-\gamma_{ij}))+I_i(t),\ i=1,2,\ldots,n,\ t\in\mathbb{R}, \tag{1.2}$$

if $\mathbb{T}=\mathbb{Z}$, *then* (1.1) *reduces to the following form*

$$\begin{aligned}x_i(k+1)-x_i(k)\ &=\ -c_i(k)x_i(t)+\sum_{j=1}^n a_{ij}(k)f_j(x_j(k))+\sum_{j=1}^n b_{ij}(t)f_j(x_j(k-\gamma_{ij}))\\&\quad+I_i(k),i=1,2,\ldots,n,\ k\in\mathbb{Z}.\end{aligned} \tag{1.3}$$

To the best of our knowledge, there is no paper published on the existence and exponential stability of weighted pseudo-almost periodic solutions for (1.2) *and* (1.3)*.*

The organization of the rest of this paper is as follows. In Section 2, we introduce some definitions and make some preparations for later sections. In Section 3, we propose a concept of weighted pseudo-almost periodic functions on almost periodic time scales and study some their basic properties. In Section 4, we study the existence of weighted pseudo-almost periodic

solutions to linear dynamic equations on time scales. In Section 5 and Section 6, based on the results obtained in the previous sections, Banach's fixed point theorem and Δ-differential inequalities on time scales, we present some sufficient conditions which guarantee the existence and global exponential stability of weighted pseudo-almost periodic solutions to (1.1). In Section 7, we present examples to illustrate the feasibility and effectiveness of our results obtained in Section 5 and Section 6.

2 Preliminaries

In this section, we shall first recall some basic definitions and prove some lemmas.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \ \text{ and } \ \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $Y^{\Delta}(t) = y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s = Y(t) - Y(a)$.

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and right-dense continuous functions $r : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{r \in \mathcal{R} : 1 + \mu(t)r(t) > 0, \ \forall t \in \mathbb{T}\}$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}, \ \text{ for } s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Then the generalized exponential function has the following properties.

Lemma 2.1. *[37, 38] Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(s)p(s)}$;

(iv) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(v) $(e_{\ominus p}(t,s))^{\Delta} = (\ominus p)(t) e_{\ominus p}(t,s)$.

Definition 2.1. *[39] For every $x, y \in \mathbb{R}$, $[x, y) = \{t \in \mathbb{R} : x \le t < y\}$, define a countably additive measure m_1 on the set $\mathfrak{F}_1 = \{[\tilde{a}, \tilde{b}) \cap \mathbb{T} : \tilde{a}, \tilde{b} \in \mathbb{T}, \tilde{a} \le \tilde{b}\}$, that assigns to each interval $[\tilde{a}, \tilde{b}) \cap \mathbb{T}$ its length, that is $m_1([\tilde{a}, \tilde{b}) = \tilde{b} - \tilde{a}$. The interval $[\tilde{a}, \tilde{a})$ is understood as the empty set. Using m_1, it generates the outer measure m_1^* on $P(\mathbb{T})$, defined for each $E \in P(\mathbb{T})$ as*

$$m_1^*(E) = \begin{cases} \inf_{\tilde{\mathfrak{R}}} \left\{ \sum_{i \in I_{\tilde{\mathfrak{R}}}} (\tilde{b_i} - \tilde{a}_i) \right\} \in \mathbb{R}^+, & b \in \mathbb{T} \setminus E, \\ +\infty, & b \in E, \end{cases}$$

with

$$\tilde{\mathfrak{R}} = \left\{ \{[\tilde{a}_i, \tilde{b}_i) \cap \mathbb{T} \in \mathfrak{F}_1\}_{i \in I_{\tilde{\mathfrak{R}}}} : I_{\tilde{\mathfrak{R}}} \subset \mathbb{N}, E \subset \bigcup_{i \in I_{\tilde{\mathfrak{R}}}} ([a_i, b_i) \cap \mathbb{T}) \right\}.$$

A set $\wedge \subset \mathbb{T}$ is said to be Δ-measurable if the following equality:

$$m_1^*(E) = m_1^*(E \cap \wedge) + m_1^*(E \cap (\mathbb{T} \backslash \wedge))$$

holds true for all subset E of $\mathbb{T}$. Define the family $\mathcal{M}(m_1^*) = \{\wedge \subset \mathbb{T} : \wedge$ is Δ-measurable$\}$, the Lebesgue Δ-measure, denoted by μ_Δ is the restriction of m_1^* to $\mathcal{M}(m_1^*)$.

Definition 2.2. *[39] We say that $f : \mathbb{T} \to \bar{\mathbb{R}} \equiv [-\infty, +\infty]$ is Δ-measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ is Δ-measurable.*

Lemma 2.2. *[39] Let $\wedge \subset \mathbb{T}$. Then $\wedge$ is Δ-measurable if and only if $\wedge$ is Lebesgue measurable.*

By using Lemma 2.2, we can get the following two corollaries:

Corollary 2.1. *If A is a closed subset of $\mathbb{R}$, then $A \cap \mathbb{T}$ is Δ-measurable.*

Corollary 2.2. *If $f \in C(\mathbb{T}, \mathbb{R})$, then f is Δ-measurable.*

Theorem 2.1. *[39] Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of Δ-measurable functions such that for every $t \in \mathbb{T}$ the following conditions are satisfied:*

(a) $0 \le f_m(t) \le f_{m+1}(t) \le \infty$ *for all* $m \in \mathbb{N}$*;*

(b) $\lim\limits_{m\to\infty} f_m(t) = f(t)$.

Then f is Δ-measurable and $\lim_{m\to\infty}\int_E f_m(s)\Delta s = \int_E f(s)\Delta s$.

Definition 2.3. *[14, 15] A time scale is called an almost periodic time scale if*

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

In this paper, we restrict our discussions on almost periodic time scales.

Lemma 2.3. *[40] If $\mathbb{T}$ is an almost periodic time scale and $\tau \in \Pi$, then $\sigma(t+\tau) = \sigma(t) + \tau$ for $t \in \mathbb{T}$.*

Corollary 2.3. *If $\mathbb{T}$ is an almost periodic time scale, then $\mu(t+\tau) = \mu(t)$, $\forall t \in \mathbb{T}$, $\tau \in \Pi$.*

Lemma 2.4. *Let $\mathbb{T}$ be an almost periodic time scale. If $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous, then*

$$\int_a^b f(t+\tau)\Delta t = \int_{a+\tau}^{b+\tau} f(t)\Delta t,$$

where $a, b \in \mathbb{T}$, $\tau \in \Pi$.

Proof. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t) := F(t+\tau)$ for all $t \in \mathbb{T}$. Since t is right-scattered or right-dense if and only if $t+\tau$ are right-scattered or right-dense, respectively, $G(t)$ is a continuous function.

Case (1): If t is right-scattered, then $G(t)$ is differentiable at t and

$$G^{\Delta}(t) = \frac{G(\sigma(t)) - G(t)}{\mu(t)} = \frac{F(\sigma(t)+\tau) - F(t+\tau)}{\mu(t+\tau)} = \frac{F(\sigma(t+\tau)) - F(t+\tau)}{\mu(t+\tau)} = f(t+\tau).$$

Case (2): If t is right-dense, since F is differentiable at t, $\lim\limits_{s\to t}\frac{F(t)-F(s)}{t-s}$ exists and is a finite number. In this case $f(t) = F^{\Delta}(t) = \lim\limits_{s\to t}\frac{F(t)-F(s)}{t-s}$, so

$$f(t+\tau) = \lim_{s'\to t+\tau}\frac{F(t+\tau) - F(s')}{t+\tau-s'} = \lim_{s\to t}\frac{F(t+\tau) - F(s+\tau)}{t+\tau-(s+\tau)} = \lim_{s\to t}\frac{G(t)-G(s)}{t-s} = G^{\Delta}(t).$$

Therefore, for both cases we have $G^{\Delta}(t) = f(t+\tau)$ for all $t \in \mathbb{T}$. Consequently,

$$\int_{a+\tau}^{b+\tau} f(t)\Delta t = F(b+\tau) - F(a+\tau) = G(b) - G(a) = \int_a^b f(t+\tau)\Delta t.$$

The proof is complete. ∎

Example 2.1. *Consider the time scale* $\mathbb{T} = \bigcup_{k=-\infty}^{+\infty}[2k, 2k+1]$. *Let f be a right-dense continuous function. Obviously for this time scale,* $2 \in \Pi$ *and*

$$\int_0^2 f(t+2)\Delta t = \int_0^1 f(t+2)dt + \int_1^{\sigma(1)} f(t+2)\Delta t = \int_2^3 f(t)dt + \mu(1)f(3) = \int_2^3 f(t)dt + f(3),$$

$$\int_2^4 f(t)\Delta t = \int_2^3 f(t)dt + \int_3^{\sigma(3)} f(t)\Delta t = \int_2^3 f(t)dt + \mu(3)f(3) = \int_2^3 f(t)dt + f(3).$$

Now, for convenience, we introduce some notations. We will use $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, in which the symbol T denotes the transpose of vectors. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$, and define $||x|| = \max\limits_{1\le i\le n} |x_i|$.

Let $BC(\mathbb{T}, \mathbb{R}^n) = \{f : \mathbb{T} \to \mathbb{R}^n | f$ is bounded continuous function on $\mathbb{T}\}$ with the sup-norm defined by $||f||_\infty = \sup\limits_{t\in\mathbb{T}} \|f(t)\|$. It is easy to check that $(BC(\mathbb{T}, \mathbb{R}^n), ||\cdot||_\infty)$ is a Banach space.

Definition 2.4. *[14, 15] Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called almost periodic if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains at least a number $\tau \in \Pi$ with the following property*

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{T}.$$

The collection of all almost periodic functions which go from $\mathbb{T}$ to $\mathbb{R}^n$ will be denoted by $AP(\mathbb{T}, \mathbb{R}^n)$. $AP(\mathbb{T}, \mathbb{R}^n)$ equipped with the sup-norm is a Banach space.

Definition 2.5. *[16] A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called pseudo-almost periodic if $f = g + h$, where $g \in AP(\mathbb{T}, \mathbb{R}^n)$ and $h \in PAP_0^*(\mathbb{T}, \mathbb{R}^n) := \big\{\varphi \in BC(\mathbb{T}, \mathbb{R}^n) : \varphi$ is Δ-measurable such that $\lim\limits_{r\to+\infty} \frac{1}{2r}\int_{\bar{t}-r}^{\bar{t}+r} |\varphi(s)|\Delta s = 0\big\}$, where $\bar{t} \in \mathbb{T}$, $r \in \Pi$.*

Definition 2.6. *The weighted pseudo-almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ of system (1.1) with the initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$ is said to be globally exponentially stable. If there exist a positive constant λ with $\ominus\lambda \in \mathcal{R}^+$ and $M > 1$ such that every solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of system (1.1) with the initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$ satisfies*

$$\|x(t) - x^*(t)\| \le M e_{\ominus\lambda}(t, t_0)\|\psi\|_\infty, \quad \forall t \in (0, +\infty)_{\mathbb{T}},$$

where

$$\|\psi\|_\infty = \sup_{t\in[-\gamma,0]_{\mathbb{T}}} \max_{1\le i\le n} |\varphi_i(t) - \varphi_i^*(t)|, \quad t_0 = \max\{[-v, 0]_{\mathbb{T}}\}.$$

3 Weighted pseudo-almost periodic functions on time scales

Let $\mathbb{U}$ denote the collection of functions (weights) $u : \mathbb{T} \to (0, +\infty)$, which are locally integrable over $\mathbb{T}$ such that $u > 0$ almost everywhere. Let $u \in \mathbb{U}$, for $r \in \Pi$ with $r > 0$, we denote

$$u(Q_r) := \int_{Q_r} u(x)\Delta x,$$

where $Q_r := [\bar{t}-r, \bar{t}+r]_{\mathbb{T}}$ ($\bar{t} = \min\{[0,\infty)_{\mathbb{T}}\}$). If $u(x) = 1$ for each $x \in \mathbb{T}$, then $\lim\limits_{r\to\infty} u(Q_r) = \infty$. Consequently, we define the space of weights $\mathbb{U}_\infty$ by $\mathbb{U}_\infty := \big\{u \in \mathbb{U} : \inf\limits_{t\in\mathbb{T}} u(t) = u_0 > 0, \lim\limits_{r\to\infty} u(Q_r) = \infty\big\}$. In addition to the above, we define the set of weights $\mathbb{U}_B$ by $\mathbb{U}_B := \big\{u \in \mathbb{U}_\infty : \sup\limits_{t\in\mathbb{T}} u(t) < \infty\big\}$.

Definition 3.1. *Fix $u \in \mathbb{U}_\infty$. A continuous function $f : \mathbb{T} \to \mathbb{R}^n$ is called weighted pseudo-almost periodic if it can be written as $f = h + \varphi$ with $h \in AP(\mathbb{T}, \mathbb{R}^n)$ and $\varphi \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$, where the space $PAP_0(\mathbb{T}, \mathbb{R}^n, u)$ is defined by*

$$PAP_0(\mathbb{T}, \mathbb{R}^n, u) = \left\{\varphi \in BC(\mathbb{T}, \mathbb{R}^n) : \lim_{r\to+\infty} \frac{1}{u(Q_r)} \int_{Q_r} \|g(t)\| u(t)\Delta t = 0\right\}.$$

$PAP_0(\mathbb{T}, \mathbb{R}^n, u)$ is called translation invariant if $\varphi \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$, then $\varphi_\tau \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$, where $\tau \in \Pi$ and $\varphi_\tau(t) = \varphi(t - \tau)$.

All weighted pseudo-almost periodic functions which go from $\mathbb{T}$ to $\mathbb{R}^n$, will be denoted by $PAP(\mathbb{T}, \mathbb{R}^n, u)$. By Definition 2.5, it is easy to see that the decomposition of a pseudo-almost periodic function on time scales is unique. This is not always the case for weighted pseudo-almost periodic functions (see Example 3.1). In fact, the uniqueness of such a decomposition depends upon the translation-invariance of the space $PAP_0(\mathbb{T}, \mathbb{R}^n, u)$.

Example 3.1. *Consider the almost periodic time scale $\mathbb{T} = \bigcup\limits_{k=-\infty}^{+\infty} [2k, 2k+1]$. Let $u(t) = e^{|t|}$. Obviously $\inf\limits_{t\in\mathbb{T}} u(t) = 1 > 0$, $\Pi = 2\mathbb{Z}$, and*

$$\begin{aligned}
\lim_{r\to\infty} u(Q_r) &= \lim_{k\to\infty} \int_{-2k}^{2k} e^{|t|}\Delta t \\
&= \lim_{k\to\infty} \Big[\int_{-2k}^{-2k+1} e^{-t}dt + \int_{-2k+1}^{\sigma(-2k+1)} e^{-t}\Delta t + \int_{-2k+2}^{-2k+3} e^{-t}dt \\
&\quad + \int_{-2k+3}^{\sigma(-2k+3)} e^{-t}\Delta t + \ldots + \int_{-2}^{-1} e^{-t}dt + \int_{-1}^{\sigma(-1)} e^{-t}\Delta t + \int_0^1 e^t dt \\
&\quad + \int_1^{\sigma(1)} e^t\Delta t + \int_2^3 e^t dt + \int_3^{\sigma(3)} e^t\Delta t + \ldots + \int_{2k-2}^{2k-1} e^t dt + \int_{2k-1}^{\sigma(2k-1)} e^t\Delta t\Big]
\end{aligned}$$

$$= \lim_{k\to\infty}[e^{2k} - 1 + 2(e^1 + e^3 + \ldots + e^{2k-1})] = \infty,$$

which implies that $u \in \mathbb{U}_\infty$. It is easy to see that $f(t) = (\sin(2\pi t), \sin(4\pi t))^T \in AP(\mathbb{T}, \mathbb{R}^2)$. For each $i \in \mathbb{N}$, we have

$$\int_{-2i}^{-2i+1} |\sin(2\pi t)|e^{|t|}\Delta t = \int_{-2i}^{-2i+1} |\sin(2\pi t)|e^{-t}dt = 0,$$

$$\int_{-2i+1}^{\sigma(-2i+1)} |\sin(2\pi t)|e^{|t|}\Delta t = \mu(-2i+1)|\sin(2\pi(-2i+1))|e^{|-2i+1|} = 0,$$

$$\int_{2i-2}^{2i-1} |\sin(2\pi t)|e^{|t|}\Delta t = \int_{2i-2}^{2i-1} |\sin(2\pi t)|e^{t}dt = 0,$$

$$\int_{2i-1}^{\sigma(2i-1)} |\sin(2\pi t)|e^{|t|}\Delta t = \mu(2i-1)|\sin(2\pi - 2i + 1))|e^{|2i-1|} = 0,$$

so $\frac{1}{u(Q_r)}\int_{Q_r} |\sin(2\pi t)|e^{|t|}\Delta t = 0$. Similarly, $\frac{1}{u(Q_r)}\int_{Q_r} |\sin(4\pi t)|e^{|t|}\Delta t = 0$, that is, $f(t) \in AP(\mathbb{T}, \mathbb{R}^2) \cap PAP_0(\mathbb{T}, \mathbb{R}^2, u)$.

Similar to the proof of Theorem 2.1 in [41], we have

Theorem 3.1. *Suppose that $u \in \mathbb{U}_\infty$ and for any $\tau \in \Pi$, $\overline{\lim\limits_{|t|\to+\infty}}\frac{u(t+\tau)}{u(t)}$ is finite, then $\overline{\lim\limits_{|t|\to+\infty}}\frac{u(Q_{t+\tau})}{u(Q_t)}$ is finite and $PAP_0(\mathbb{T}, \mathbb{R}^n, u)$ is translation invariant.*

Based on Theorem 3.1, we introduce the following new set of weights, which makes the spaces of weighted pseudo-almost periodic functions translation invariant:

$$\mathbb{U}_\infty^{Inv} := \left\{u \in \mathbb{U}_\infty : \text{ for all } s \in \Pi, \overline{\lim_{|t|\to\infty}}\frac{u(t+s)}{u(t)} < \infty\right\}.$$

By Definition 3.1 and the definition of $\mathbb{U}_\infty^{Inv}$, one can easily show that

Lemma 3.1. *Let $u \in \mathbb{U}_\infty^{Inv}$. If $f, g \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, then $f + g, fg \in PAP(\mathbb{T}, \mathbb{R}^n, u)$; if $f \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, $g \in AP(\mathbb{T}, \mathbb{R}^n)$, then $fg \in PAP(\mathbb{T}, \mathbb{R}^n, u)$.*

Theorem 3.2. *Let $u \in \mathbb{U}_\infty^{Inv}$. If $f \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, then there exist a unique $g \in AP(\mathbb{T}, \mathbb{R}^n)$ and a unique $h \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$ such that $f = g + h$.*

Proof. Suppose that $f \not\equiv 0$ and $f \in AP(\mathbb{T}, \mathbb{R}^n) \cap PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. Then there exists a $t_0 \in \mathbb{T}$ such that $f(t_0) \neq 0$. One can assume that there exists $\delta > 0$ such that $\|f(t_0)\| \geq 2\delta$. Define

$$B_\delta := \{\tau \in \Pi : \|f(t_0 + \tau) - f(t_0)\| \leq \delta\}$$

for every $r > 0$ and $r \in \Pi$. For any $t \in [\overline{t}-r, \overline{t}+r]_\mathbb{T}$ ($\overline{t} = \min\{[0, +\infty)_\mathbb{T}\}$), since $f \in AP(\mathbb{T}, \mathbb{R}^n)$, there exists $l_\delta > 0$ and $\tau \in [t - l_\delta, t]_\mathbb{T} \cap B_\delta$, we have

$$t = \tau + (t - \tau) \in (t - \tau) + B_\delta \subset \bigcup_{s\in\mathbb{T}}(s + B_\delta).$$

On the other hand, $[\bar{t}-r,\bar{t}+r]_{\mathbb{T}}$ is a bounded closed subset of $\mathbb{R}$, so $[\bar{t}-r,\bar{t}+r]_{\mathbb{T}}$ is a compact subset of $\mathbb{R}$, then there exist $s_1, s_2, \ldots, s_m \in \mathbb{T}$ such that

$$[\bar{t}-r,\bar{t}+r]_{\mathbb{T}} \subset \bigcup_{k=1}^{m}(s_k + B_\delta).$$

Noticing that

$$\|f(t_0+t)\| \geq \|f(t_0)\| - \|f(t_0+t) - f(t_0)\| \geq \delta$$

for all $t \in B_\delta$. For every $t \in [\bar{t}-r,\bar{t}+r]_{\mathbb{T}}$, there exists an $i \in \{1, 2, \ldots, m\}$ such that $t - s_i \in B_\delta$, hence $\|f(t - s_i + t_0)\| \geq \delta$. Set

$$F(t) = \|f(t+t_0)\| + \|f(t+t_0-s_1)\| + \|f(t+t_0-s_2)\| + \ldots + \|f(t+t_0-s_m)\|.$$

One can easily see that $F(t) \geq \delta$ for all $t \in [\bar{t}-r,\bar{t}+r]_{\mathbb{T}}$. So

$$\frac{1}{u(Q_r)} \int_{Q_r} F(t)u(t)\Delta t \geq \frac{\delta}{u(Q_r)} \int_{Q_r} u(t)\Delta t = \delta. \tag{3.1}$$

Using the fact that $PAP_0(\mathbb{T}, \mathbb{R}^n, u)$ is translation-invariant and $f \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$, it follows that $f(t+t_0), f(t+t_0-s_k)$ $(k = 1, 2, \ldots, m) \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$, that is,

$$\lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \|f(t+t_0)\|u(t)\Delta t = 0$$

and

$$\lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \|f(t+t_0-s_k)\|u(t)\Delta t = 0, \ \ k = 1, 2, \ldots, m,$$

which contradict (3.1), and hence $AP(\mathbb{T}, \mathbb{R}^n) \cap PAP_0(\mathbb{T}, \mathbb{R}^n, u) = \{0\}$, that is, $PAP(\mathbb{T}, \mathbb{R}^n, u) = AP(\mathbb{T}, \mathbb{R}^n) \bigoplus PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. The proof is complete. ∎

Lemma 3.2. *Let $u \in \mathbb{U}_\infty^{Inv}$. If $f = g + h \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, where $g \in AP(\mathbb{T}, \mathbb{R}^n)$, then $g(\mathbb{T}) \subset \overline{f(\mathbb{T})}$ and $||g||_\infty \leq ||f||_\infty$.*

Proof. If we suppose that $g(\mathbb{T}) \subset \overline{f(\mathbb{T})}$ does not hold, then exist $t_0 \in \mathbb{T}$ and $\varepsilon_0 > 0$ such that $\inf_{s\in\mathbb{T}} \|g(t_0) - f(s)\| > \varepsilon_0$. Using the continuity of the function g, there exists $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$, $\|g(t) - g(t_0)\| < \frac{\varepsilon_0}{2}$. Since $g \in AP(\mathbb{T}, \mathbb{R}^n)$, there exists $l_{\frac{\varepsilon_0}{4}} > 0$ such that every interval of length $l_{\frac{\varepsilon_0}{4}}$ contains a $\tau \in \Pi$ with the property that

$$\|g(t+\tau) - g(t)\| < \frac{\varepsilon_0}{4}, \ t \in \mathbb{T}.$$

So

$$\begin{aligned}\|h(t+\tau)\| &= \|f(t+\tau) - g(t+\tau)\| \\ &\geq \|f(t+\tau) - g(t)\| - \|g(t) - g(t+\tau)\|\end{aligned}$$

$$\begin{aligned}&\geq \|f(t+\tau)-g(t_0)\|-\|g(t_0)-g(t)\|-\|g(t)-g(t+\tau)\|\\&>\frac{\varepsilon_0}{4},\ \forall t\in(t_0-\delta,t_0+\delta)\cap\mathbb{T},\end{aligned}$$

which implies that

$$\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\|h(t+\tau)\|u(t)\Delta t\geq\frac{\varepsilon_0}{4}$$

and this is a contradiction. The proof is complete. ∎

Lemma 3.3. *Let $u\in\mathbb{U}_\infty^{Inv}$. If $(f_m)_{m\in\mathbb{N}}\subset APA(\mathbb{T},\mathbb{R}^n,u)$ such that $\lim\limits_{m\to+\infty}||f_m-f||_\infty=0$, then $f\in PAP(\mathbb{T},\mathbb{R}^n,u)$.*

Proof. Since $(f_m)_{m\in\mathbb{N}}\subset PAP(\mathbb{T},\mathbb{R}^n,u)$, there exist $(g_m)_{m\in\mathbb{N}}\subset AP(\mathbb{T},\mathbb{R}^n)$ and $(h_m)_{m\in\mathbb{N}}\subset PAP_0(\mathbb{T},\mathbb{R}^n,u)$ such that $f_m=g_m+h_m$, $\forall m\in\mathbb{N}$. According to Lemma 3.1 and Theorem 3.2, we have $f_s-f_m=(g_s-g_m)+(h_s-h_m)\in PAP(\mathbb{T},\mathbb{R}^n,u), g_s-g_m\in AP(\mathbb{T},\mathbb{R}^n)$ and $h_s-h_m\in PAP_0(\mathbb{T},\mathbb{R}^n,u)$. From Lemma 3.2 it follows that $||g_s-g_m||_\infty\leq||f_s-f_m||_\infty$. Since $\lim\limits_{m\to\infty}||f_m-f||_\infty=0$ it follows that $(f_m)_{m\in\mathbb{N}}$ is a cauchy sequence, and $||g_s-g_m||_\infty\to0$ as $s,m\to+\infty$ too, that is, $(g_m)_{m\in\mathbb{N}}$ is a cauchy sequence. Using the fact that $(AP(\mathbb{T},\mathbb{R}^n),||\cdot||_\infty)$ is a Banach space, it follows that there exists $g\in AP(\mathbb{T},\mathbb{R}^n)$ such that $\lim\limits_{m\to\infty}||g_m-g||_\infty=0$. Let $h=f-g$, it is easy to see that $\lim\limits_{m\to\infty}||h_m-h||_\infty=0$, which yields that $h\in BC(\mathbb{T},\mathbb{R}^n)$. On the other hand,

$$\begin{aligned}\frac{1}{u(Q_r)}\int_{Q_r}||h(t)||u(t)\Delta t&=\frac{1}{u(Q_r)}\int_{Q_r}||h(t)-h_m(t)+h_m(t)||u(t)\Delta t\\&\leq\frac{1}{u(Q_r)}\int_{Q_r}||h(t)-h_m(t)||u(t)\Delta t+\frac{1}{u(Q_r)}\int_{Q-r}||h_m(t)||u(t)\Delta t\\&\leq||h_m-h||_\infty+\frac{1}{u(Q_r)}\int_{Q_r}||h_m(t)||u(t)\Delta t.\end{aligned}$$

Letting $r\to\infty$, it follows that

$$\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}||h(t)||u(t)\Delta t\leq||h_m-h||_\infty.$$

Letting $m\to\infty$ in the previous inequality, we get

$$\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}||h(t)||u(t)\Delta t=0,$$

that is, $h\in PAP_0(\mathbb{T},\mathbb{R}^n,u)$. The proof is complete. ∎

Corollary 3.1. *Let $u\in\mathbb{U}_\infty^{Inv}$. Then $(PAP(\mathbb{T},\mathbb{R}^n,u),||\cdot||_\infty)$ is a Banach space.*

Proof. By Lemma 3.3, $PAP(\mathbb{T},\mathbb{R}^n,u)$ is closed and $PAP(\mathbb{T},\mathbb{R}^n,u)\subset BC(\mathbb{T},\mathbb{R}^n)$. Therefore, $(PAP(\mathbb{T},\mathbb{R}^n,u),||\cdot||_\infty)$ is a Banach space. The proof is complete. ∎

Lemma 3.4. *Let $u \in \mathbb{U}_\infty^{Inv}$. If $f : \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition and $\varphi \in PAP(\mathbb{T}, \mathbb{R}, u), \tau \in \Pi$, then $\Gamma : t \to f(\varphi(t-\tau))$ belongs to $PAP(\mathbb{T}, \mathbb{R}, u)$.*

Proof. Since $\varphi \in PAP(\mathbb{T}, \mathbb{R}, u)$, there exist $\varphi_1 \in AP(\mathbb{T}, \mathbb{R})$ and $\varphi_2 \in PAP_0(\mathbb{T}, \mathbb{R}, u)$ such that $\varphi = \varphi_1 + \varphi_2$. Set

$$\Gamma(t) = f(\varphi(t-\tau)) = f(\varphi_1(t-\tau)) + [f(\varphi_1(t-\tau) + \varphi_2(t-\tau)) - f(\varphi_1(t-\tau))] := \Gamma_1(t) + \Gamma_2(t).$$

First, we prove that $\Gamma_1 \in AP(\mathbb{T}, \mathbb{R})$. Since f satisfies the Lipschitz condition, there exists a positive constant L such that $|f(u_1) - f(u_2)| \le L|u_1 - u_2|$, $\forall u_1, u_2 \in \mathbb{R}$. For any $\varepsilon > 0$, since $\varphi_1 \in AP(\mathbb{T}, \mathbb{R})$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\alpha = \alpha(\varepsilon) \in \Pi$ in this interval such that $|\varphi_1(t+\alpha) - \varphi_1(t)| < \frac{\varepsilon}{L}$ for $\forall t \in \mathbb{T}$, then

$$|\Gamma_1(t+\alpha) - \Gamma_1(t)| = |f(\varphi_1(t+\alpha-\tau)) - f(\varphi_1(t-\tau))| \le L|\varphi_1(t+\alpha-\tau) - \varphi_1(t-\tau)| < \varepsilon,$$

which implies that $\Gamma_1 \in AP(\mathbb{T}, \mathbb{R})$. Next we prove that $\Gamma_2 \in PAP_0(\mathbb{T}, \mathbb{R}, u)$. Since $\varphi_2 \in PAP_0(\mathbb{T}, \mathbb{R}, u)$, by using Theorem 3.1, we have $\varphi_2(t-\tau) \in PAP_0(\mathbb{T}, \mathbb{R}, u)$, so

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} |\Gamma_2(t)| u(t) \Delta t &= \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} |f(\varphi_1(t-\tau) + \varphi_2(t-\tau)) - f(\varphi_1(t-\tau))| u(t) \Delta t \\ &\le \lim_{r\to\infty} \frac{L}{u(Q_r)} \int_{Q_r} |\varphi_2(t-\tau)| u(t) \Delta t = 0,\end{aligned}$$

which implies that $\Gamma_2 \in PAP_0(\mathbb{T}, \mathbb{R}, u)$. Consequently, $\Gamma \in PAP(\mathbb{T}, \mathbb{R}, u)$. The proof is complete. ∎

4 Weighted pseudo-almost periodic solutions of linear dynamic equations on time scales

Consider the non-autonomous equation

$$x^\Delta = A(t)x + F(t) \tag{4.1}$$

and its associated homogeneous equation

$$x^\Delta = A(t)x, \tag{4.2}$$

where the $n \times n$ coefficient matrix $A(t)$ is continuous on $\mathbb{T}$ and the column vector $F = (f_1, f_2, \ldots, f_n)^T : \mathbb{T} \to \mathbb{R}^n$. Define $\|F\| = \sup_{t\in\mathbb{T}} \|F(t)\|$. We will call $A(t)$ is almost periodic if all of its entries are almost periodic.

Definition 4.1. *[14] Equation (4.2) is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (4.2), satisfying*

$$\|X(t)PX^{-1}(\sigma(s))\|_0 \leq ke_{\ominus\alpha}(t, \sigma(s)), \ \ s, t \in \mathbb{T}, \ t \geq \sigma(s),$$
$$\|X(t)(I-P)X^{-1}(\sigma(s))\|_0 \leq ke_{\ominus\alpha}(\sigma(s), t), \ \ s, t \in \mathbb{T}, \ t \leq \sigma(s),$$

where $\|\cdot\|_0$ is a matrix norm on $\mathbb{T}$.

Lemma 4.1. *Suppose $a > 0$, then*

$$e_{\ominus a}(t, s) \leq \exp\left(\frac{-a}{1+\overline{\mu}a}(t-s)\right), \ \forall s \leq t,$$

where $\overline{\mu} = \sup\limits_{t\in\mathbb{T}} \mu(t)$.

Proof. For every $\tau \in \mathbb{T}$, if $\mu(\tau) = 0$, then

$$\xi_{\mu(\tau)}(\ominus a) = \frac{-a}{1+\mu(\tau)a} = -a \leq \frac{-a}{1+1+\overline{\mu}a};$$

if $\mu(\tau) > 0$, then

$$\begin{aligned}\xi_{\mu(\tau)}(\ominus a) &= \frac{\log(1+\mu(\tau)\ominus a)}{\mu(\tau)} = \frac{\log(1-\mu(\tau)\frac{a}{1+\mu(\tau)a})}{\mu(\tau)} = \frac{-\log(1+\mu(\tau)a)}{\mu(\tau)} \\ &\leq \frac{\frac{-\mu(\tau)a}{1+\mu(\tau)a}}{\mu(\tau)} = \frac{-a}{1+\mu(\tau)a} \leq \frac{-a}{1+\overline{\mu}a}.\end{aligned}$$

Thus, we have

$$\xi_{\mu(\tau)}(\ominus a) \leq \frac{-a}{1+\overline{\mu}a}, \ \ \forall \tau \in \mathbb{T},$$

so

$$e_{\ominus a}(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(\ominus a)\Delta\tau\right) \leq \exp\left(\int_s^t \frac{-a}{1+\overline{\mu}a}\Delta\tau\right) = \exp\left(\frac{-a}{1+\overline{\mu}a}(t-s)\right).$$

The proof is complete. ∎

Lemma 4.2. *[15] Let $c_i(t)$ be an almost periodic function on $\mathbb{T}$, where $c_i(t) > 0, -c_i(t) \in \mathcal{R}^+, i = 1, 2, \ldots, n, \forall t \in \mathbb{T}$ and $\min\limits_{1\leq i\leq n}\{\inf\limits_{t\in\mathbb{T}} c_i(t)\} = \widetilde{m} > 0$, then the linear system*

$$x^{\Delta}(t) = \operatorname{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

Theorem 4.1. *Let $u \in \mathbb{U}_\infty^{Inv}$. Assume that $A(t)$ is almost periodic, (4.2) admits an exponential dichotomy and the function $F \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. Then (4.1) has a unique bounded solution $x \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$.*

Proof. Similar to the proof of Theorem 5.1 in [16], we have

$$x(t)=\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))F(s)\Delta s-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))F(s)\Delta s$$

is a unique bounded solution of (4.1). Next, we will show that $x\in PAP_0(\mathbb{T},\mathbb{R}^n,u)$. Let

$$I(t)=\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))F(s)\Delta s,\ H(t)=\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))F(s)\Delta s.$$

By using Lemma 2.3, Lemma 4.1 and in view of Definition 4.1, we can get

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\|I(t)\|u(t)\Delta t &= \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big\|\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))F(s)\Delta s\Big\|u(t)\Delta t\\
&\le \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big(\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|\|F(s)\|\Delta s\Big)u(t)\Delta t\\
&\le \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big(\int_{-\infty}^{t}Ke_{\ominus\alpha}(t,\sigma(s))\|F(s)\|\Delta s\Big)u(t)\Delta t\\
&\le \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big(\int_{-\infty}^{t}Ke^{-\frac{\alpha}{1+\overline{u}\alpha}(t-\sigma(s))}\|F(s)\|\Delta s\Big)u(t)\Delta t\\
&\le \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big(\int_{-\infty}^{t}Ke^{-\frac{\alpha}{1+\overline{u}\alpha}(t-s-k)}\|F(s)\|\Delta s\Big)u(t)\Delta t\\
&= \lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\Big(\int_{-k}^{+\infty}Ke^{-\frac{\alpha}{1+\overline{u}\alpha}s}\|F(t-s-k)\|\Delta s\Big)u(t)\Delta t\\
&= \lim_{r\to\infty}\int_{-k}^{+\infty}Ke^{-\frac{\alpha}{1+\overline{u}\alpha}s}\Big(\frac{1}{u(Q_r)}\int_{Q_r}\|F(t-s-k)\|u(t)\Delta t\Big)\Delta s,\end{aligned}$$

where $k(>0)\in\Pi$. Consider the following function

$$\Gamma_r(s)=\frac{1}{u(Q_r)}\int_{Q_r}\|F(t-s-k)\|\mu(t)\Delta t.$$

Obviously, $\Gamma_r(s)$ is bounded. By using Corollary 2.2, we have that $\Gamma_r(s)$ is Δ-measurable and by using Theorem 3.1, we can get $\lim_{r\to\infty}\Gamma_r(s)=0$. Consequently, by Theorem 2.1, we obtain

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\|I(t)\|u(t)\Delta t &= \lim_{r\to\infty}\int_{-k}^{+\infty}Ke^{-\frac{\alpha}{1+\overline{u}\alpha}s}\Gamma_r(s)\Delta s\\
&= \int_{-k}^{+\infty}\lim_{r\to\infty}(Ke^{-\frac{\alpha}{1+\overline{u}\alpha}s}\Gamma_r(s))\Delta s=0. \qquad (4.3)\end{aligned}$$

By using Lemma 2.3, Lemma 4.1 and in view of Definition 4.1, we can obtain

$$\lim_{r\to\infty}\frac{1}{u(Q_r)}\int_{Q_r}\|H(t)\|u(t)\Delta t$$

$$
\begin{aligned}
&= \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \Big\| \int_t^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))F(s)\Delta s \Big\| u(t)\Delta t \\
&\le \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \Big(\int_t^{+\infty} \|X(t)(I-P)X^{-1}(\sigma(s))\| \|F(s)\|\Delta s \Big) u(t)\Delta t \\
&\le \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \big(K e_{\ominus\alpha}(\sigma(s),t)\|F(s)\|\Delta s \big) u(t)\Delta t \\
&\le \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \Big(\int_t^{+\infty} K e^{-\frac{\alpha}{1+\overline{u}\alpha}(\sigma(s)-t)} \|F(s)\|\Delta s \Big) u(t)\Delta t \\
&\le \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \Big(\int_t^{+\infty} K e^{-\frac{\alpha}{1+\overline{u}\alpha}(s-t)} \|F(s)\|\Delta s \Big) u(t)\Delta t \\
&= \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \Big(\int_0^{+\infty} K e^{-\frac{\alpha}{1+\overline{u}\alpha}s} \|F(s+t)\|\Delta s \Big) u(t)\Delta t \\
&= \lim_{r\to\infty} \int_0^{+\infty} K e^{-\frac{\alpha}{1+\overline{u}\alpha}s} \Big(\frac{1}{u(Q_r)} \int_{Q_r} \|F(s+t\|u(t)\Delta t \Big) \Delta s.
\end{aligned}
$$

Let

$$
T_r(s) = \frac{1}{u(Q_r)} \int_{Q_r} \|F(s+t)\|u(t)\Delta t.
$$

It is easy to see that $T_r(s)$ is bounded. By using Corollary 2.2, we see that $T_r(s)$ is Δ-measurable and by using Theorem 3.1, we have $\lim_{r\to\infty} T_r(s) = 0$. Consequently, by Theorem 2.1, we get

$$
\begin{aligned}
\lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \|H(t)\|u(t)\Delta t &= \lim_{r\to\infty} \int_0^{+\infty} K e^{-\frac{\alpha}{1+\overline{u}\alpha}s} T_r(s)\Delta s \\
&= \int_0^{+\infty} \lim_{r\to\infty} (K e^{-\frac{\alpha}{1+\overline{u}\alpha}s} T_r(s))\Delta s = 0. \qquad (4.4)
\end{aligned}
$$

From (4.3) and (4.4), we have

$$
\lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} \|x(t)\|u(t)\Delta t \le \lim_{r\to\infty} \frac{1}{u(Q_r)} \int_{Q_r} (\|I(t)\| + \|H(t)\|)u(t)\Delta t = 0,
$$

which implies that $x(t) \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. The proof is complete. ∎

Theorem 4.2. *Let $u \in \mathbb{U}_\infty^{Inv}$. Suppose that $A(t)$ is almost periodic and (4.2) admits an exponential dichotomy. Then for every $F \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, (4.1) has a unique bounded solution $x_F \in PAP(\mathbb{T}, \mathbb{R}^n, u)$.*

Proof. Since $F \in PAP(\mathbb{T}, \mathbb{R}^n, u)$, $F = G + H$ where $G \in AP(\mathbb{T}, \mathbb{R}^n)$ and $H \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. According to the proof of Theorem 4.1, the function

$$
x_F = \int_{-\infty}^t X(t)PX^{-1}(\sigma(s))F(s)\Delta s - \int_t^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))F(s)\Delta s
$$

$$
\begin{aligned}
&= \Big(\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))G(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))G(s)\Delta s\Big) \\
&\quad +\Big(\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))H(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))H(s)\Delta s\Big) \\
&:= x_G + x_H
\end{aligned}
$$

is the unique solution of (4.1), where

$$
x_G := \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))G(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))G(s)\Delta s,
$$

$$
x_H := \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))H(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))H(s)\Delta s.
$$

By Theorem 4.1 in [14], $x_G \in AP(\mathbb{T}, \mathbb{R}^n)$. By Theorem 4.1, $x_H \in PAP_0(\mathbb{T}, \mathbb{R}^n, u)$. Therefore, $x_F \in PAP(\mathbb{T}, \mathbb{R}^n, u)$. This completes the proof. ∎

5 Existence of weighted pseudo-almost periodic solutions of cellular neural networks on time scales

In this section, we will use Theorem 4.2 to study the existence of weighted pseudo-almost periodic solutions of system (1.1).

Theorem 5.1. *Let $u \in \mathbb{U}_\infty^{Inv}$. Assume that (H_1)-(H_4) and*

(H_5) $\min_{1\le i\le n}\{\Pi_i\} < \min_{1\le i\le n}\{\underline{c_i}\}$ *and there exists a constant r_0 such that* $\max_{1\le i\le n}\big\{\frac{\eta_i}{\underline{c_i}}\big\} + L \le r_0$, *where*

$$
\eta_i = \sum_{j=1}^{n} (\overline{a_{ij}} + \overline{b_{ij}})(|f_j(0)| + \alpha_j r_0), \quad \Pi_i = \sum_{j=1}^{n} (\overline{a_{ij}} + \overline{b_{ij}})\alpha_j, \quad L = \max_{1\le i\le n}\Big\{\frac{\overline{I_i}}{\underline{c_i}}\Big\},
$$

$$
\underline{c_i} = \inf_{t\in\mathbb{T}} c_i(t), \quad \overline{c_i} = \sup_{t\in\mathbb{T}} c_i(t), \quad \overline{I_i} = \sup_{t\in\mathbb{T}} |I_i(t)|, \quad i = 1, 2, \ldots, n
$$

hold, then system (1.1) *has a unique weighted pseudo-almost periodic solution in the region*

$$
E = \{\varphi \in PAP(\mathbb{T}, \mathbb{R}^n, u) : \|\varphi\|_\infty \le r_0\}.
$$

Proof. For any given $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \in E$, consider the following equation

$$
x_i^\Delta(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t)) + \sum_{j=1}^{n} b_{ij}(t) f_j(\varphi_j(t-\gamma_{ij})) + I_i(t),\ i = 1, 2, \ldots, n \quad (5.1)
$$

and its associated homogeneous equation

$$x_i^{\Delta}(t) = -c_i(t)x_i(t), \;\; i = 1, 2, \ldots, n. \tag{5.2}$$

It follows from (H_3) and Lemma 4.2 that (5.2) admits an exponential dichotomy. By Lemma 3.4, we have

$$F(t) := (F_1(t), F_2(t), \ldots, F_n(t))^T \in PAP(\mathbb{T}, \mathbb{R}^n, u),$$

where

$$F_i(t) = \sum_{j=1}^{n} a_{ij}(t)f_j(\varphi_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(\varphi_j(t-\gamma_{ij})) + I_i(t), \;\; i = 1, 2, \ldots, n.$$

By Theorem 4.2, we know that system (5.1) has exactly one weighted pseudo-almost periodic solution

$$x_\varphi(t) = \int_{-\infty}^{t} X(t)X^{-1}(\sigma(s))F(s)\Delta s = (x_{\varphi_1}(t), \ldots, x_{\varphi_n}(t))^T,$$

where

$$x_{\varphi_i}(t) = \int_{-\infty}^{t} e_{-c_i}(t, \sigma(s))F_i(s)\Delta s, \;\; i = 1, 2, \ldots, n.$$

Define a nonlinear operator on E by

$$\Phi(\varphi)(t) = (x_{\varphi_1}(t), \ldots, x_{\varphi_n}(t))^T, \;\; \forall \varphi \in E.$$

For any given $\varphi \in E$, by conditions $(H_1) - (H_5)$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{T}} |x_{\varphi_i}(t)| &= \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s)) \left(\sum_{j=1}^{n} (a_{ij}(s)f_j(\varphi_j(s)) + b_{ij}(s)f_j(\varphi_j(s-\gamma_{ij}))) + I_i(s) \right) \Delta s \right| \\
&\le \sup_{t\in\mathbb{T}} \left\{ \left| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s)) \left(\sum_{j=1}^{n} (\overline{a_{ij}} f_j(\varphi_j(s)) + \overline{b_{ij}} f_j(\varphi_j(s-\gamma_{ij}))) \right) \Delta s \right| \right\} + \frac{\overline{I_i}}{\underline{c_i}} \\
&\le \sup_{t\in\mathbb{T}} \left\{ \left| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s)) \left(\sum_{j=1}^{n} \overline{a_{ij}} \big(|f_j(0)| + \alpha_j|\varphi_j(s)|\big) \right.\right.\right. \\
&\quad \left.\left.\left. + \sum_{j=1}^{n} \overline{b_{ij}} \big(|f_j(0)| + \alpha_j|\varphi_j(s-\gamma_{ij})|\big) \right) \Delta s \right| \right\} + \frac{\overline{I_i}}{\underline{c_i}} \\
&\le \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s)) \left(\sum_{j=1}^{n} \left[\overline{a_{ij}} \big(|f_j(0)| + \alpha_j r_0\big) + \overline{b_{ij}} \big(|f_j(0)| + \alpha_j r_0\big) \right] \right) \Delta s \right| + \frac{\overline{I_i}}{\underline{c_i}} \\
&\le \frac{\eta_i}{\underline{c_i}} + L_1 \le r_0, \;\; i = 1, 2, \ldots, n.
\end{aligned}$$

Hence $\|\Phi(\varphi)\|_\infty = \max_{1\le i\le n} \sup_{t\in\mathbb{T}} |x_{\varphi_i}(t)| \le r_0$. Therefore, $\Phi(E) \subset E$.

Taking $\varphi, \psi \in E$ and combining conditions (H_1) and (H_5), we obtain that

$$
\begin{aligned}
&\sup_{t\in\mathbb{T}} |x_{\varphi_i}(t) - x_{\psi_i}(t)| \\
= &\sup_{t\in\mathbb{T}} \bigg\{\bigg| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s))\bigg(\sum_{j=1}^{n} a_{ij}(s)\Big[f_j(\varphi_j(s)) - f_j(\psi_j(s))\Big] \\
&+ \sum_{j=1}^{n} b_{ij}(s)\Big[f_j(\varphi_j(s-\gamma_{ij})) - f_j(\psi_j(s-\gamma_{ij}))\Big]\bigg)\Delta s\bigg|\bigg\} \\
\leq &\sup_{t\in\mathbb{T}} \bigg\{\bigg| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s))\bigg(\sum_{j=1}^{n} a_{ij}(s)\alpha_j|\varphi_j(s) - \psi_j(s)| \\
&+ \sum_{j=1}^{n} b_{ij}(s)\alpha_j|\varphi_j(s-\gamma_{ij}) - \psi_j(s-\gamma_{ij})|\bigg)\Delta s\bigg|\bigg\} \\
\leq &\sup_{t\in\mathbb{T}} \bigg\{\bigg| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s))\bigg(\sum_{j=1}^{n} (\overline{a_{ij}} + \overline{b_{ij}})\alpha_j\bigg)\Delta s\bigg|\bigg\}\|\varphi - \psi\|_\infty \\
\leq &\frac{\Pi_i}{\underline{c_i}}\|\varphi - \psi\|_\infty < \|\varphi - \psi\|_\infty,\ i = 1, 2, \ldots, n.
\end{aligned}
\tag{5.3}
$$

From (5.3), we obtain

$$
\|\Phi(\varphi) - \Phi(\psi)\|_\infty = \max_{1\leq i\leq n} \sup_{t\in\mathbb{T}} \|x_{\varphi_i}(t) - x_{\psi_i}(t)\|_\infty < \|\varphi - \psi\|_\infty. \tag{5.4}
$$

By (5.4), we see that Φ is a contraction mapping from E to E. Since E is a closed subset of $PAP(\mathbb{T}, \mathbb{R}^n, u)$, Φ has a fixed point in E, which means that system (1.1) has a unique weighted pseudo-almost periodic solution in the region $E = \{\varphi \in PAP(\mathbb{T}, \mathbb{R}^n, u) : \|\varphi\|_\infty \leq r_0\}$. This completes the proof. ∎

Corollary 5.1. *If conditions (H_1)-(H_3) and (H_5) hold. Furthermore, assume that $I_i (i = 1, 2, \ldots, n)$ are almost periodic functions, then system* (1.1) *has a unique almost periodic solution in the region $E = \{\varphi \in AP(\mathbb{T}, \mathbb{R}^n) : ||\varphi||_\infty \leq r_0\}$.*

Corollary 5.2. *If conditions (H_1)-(H_3) and (H_5) hold. Furthermore, assume that $I_i (i = 1, 2, \ldots, n)$ are pseudo-almost periodic functions, then system* (1.1) *has a unique pseudo-almost periodic solution in the region $E = \{\varphi \in PAP(\mathbb{T}, \mathbb{R}^n) : ||\varphi||_\infty \leq r_0\}$.*

6 Exponential stability of the weighted pseudo-almost periodic solution of cellular neural networks on time scales

In this section, we derive sufficient conditions for the exponential stability of weighted pseudo-almost periodic solutions of (1.1).

Theorem 6.1. *Suppose that (H_1)-(H_5) hold, then system* (1.1) *has a unique weighted pseudo-almost periodic solution which is globally exponential stable.*

Proof. According to Theorem 3.1, for $u \in \mathbb{U}_{\infty}^{Inv}$, we know that (1.1) has a weighted pseudo-almost periodic solution $x^*(t) = \big(x_1^*(t), x_2^*(t), \ldots, x_n^*(t)\big)^T$ with the initial value $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$. Suppose that $x(t) = \big(x_1(t), x_2(t), \ldots, x_n(t)\big)^T$ is an arbitrary solution of (1.1) with the initial value $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$. Then it follows from system (1.1) that

$$\begin{aligned} & u_i^{\Delta}(s) + c_i(s)u_i(s) \\ = & \sum_{j=1}^{n} a_{ij}(s)\Big[f_j(u_j(s) + y_j^*(s)) - f_j(y_j^*(s))\Big] \\ & + \sum_{j=1}^{n} b_{ij}(s)\Big[f_j(u_j(s-\gamma_{ij}) + y_j^*(s-\gamma_{ij})) - f_j(y_j^*(s-\gamma_{ij}))\Big], \end{aligned} \tag{6.1}$$

where $u_i(s) = x_i(s) - x_i^*(s)$ and $i = 1, 2, \ldots, n$, the initial condition of (6.1) are

$$\psi_i(s) = \varphi_i(s) - \varphi_i^*(s), \quad s \in [-\gamma, 0]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.$$

Let H_i and $\overline{H_j}$ be defined by

$$H_i(\epsilon) = \underline{c_i} - \epsilon - \sum_{j=1}^{n} \alpha_j \big(\overline{a_{ij}} \exp(\epsilon \sup_{s\in\mathbb{T}} \mu(s)) + \overline{b_{ij}} \exp\big(\epsilon(\gamma + \sup_{s\in\mathbb{T}} \mu(s))\big)\big),$$

where $i = 1, 2, \ldots, n, \epsilon \in [0, +\infty)$. By (H_5), we get

$$H_i(0) = \underline{a_i} - \sum_{j=1}^{n} (\overline{a_{ij}} + \overline{b_{ij}})\alpha_j = \underline{c_i} - \Pi_i > 0, \quad i = 1, 2, \ldots, n.$$

Since $H_i, i = 1, 2, \ldots, n$ are continuous on $[0, +\infty)$ and $H_i(\epsilon) \to -\infty$ as $\epsilon \to +\infty$, there exist $\epsilon_i > 0$ such that $H_i(\epsilon_i) = 0$ and $H_i(\epsilon) > 0$ for $\epsilon \in (0, \epsilon_i)$. By choosing $\varepsilon = \min\limits_{1\le i\le n}\{\epsilon_i\}$, we have $H_i(\varepsilon) \geq 0, i = 1, 2, \ldots, n$. So, we can choose a positive constant $0 < \lambda < \min\big\{\varepsilon, \min\limits_{1\le i\le n}\{\underline{c_i}\}\big\}$ such that $H_i(\lambda) > 0, i = 1, 2, \ldots, n$, which implies that

$$\frac{1}{\underline{c_i} - \lambda}\Big[\sum_{j=1}^{n} \overline{a_{ij}}\alpha_j \exp(\lambda \sup_{s\in\mathbb{T}} \mu(s)) + \sum_{j=1}^{n} \overline{b_{ij}}\alpha_j \exp\big(\lambda(\gamma + \sup_{s\in\mathbb{T}} \mu(s))\big)\Big] < 1, \quad i = 1, 2, \ldots, n. \tag{6.2}$$

Multiplying (6.1) by $e_{-c_i}(t_0, \sigma(s))$ and integrating on $[t_0, t]_{\mathbb{T}}$, for $i = 1, 2, \ldots, n$, we obtain

$$u_i(t) = u_i(t_0)e_{-c_i}(t, t_0) + \int_{t_0}^{t} e_{-c_i}(t, \sigma(s))\bigg(\sum_{j=1}^{n} a_{ij}(s)\Big[f_j(u_j(s) + y_j^*(s)) - f_j(y_j^*(s))\Big]$$

$$+\sum_{j=1}^{n} b_{ij}(s)\bigg[f_j(u_j(s-\gamma_{ij})+y_j^*(s-\gamma_{ij}))-f_j(y_j^*(s-\gamma_{ij}))\bigg]\bigg)\Delta s. \tag{6.3}$$

Let $M=\max\limits_{1\le i\le n}\Big\{\frac{\underline{c_i}}{\sum\limits_{j=1}^{n}(\overline{a_{ij}}+\overline{b_{ij}})\alpha_j}\Big\}$, by (H_5) we have $M>1$. Thus

$$\frac{1}{M}-\frac{1}{\underline{c_i}-\lambda}\bigg[\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j\exp\big(\lambda(\gamma+\sup_{s\in\mathbb{T}}\mu(s))\big)\bigg]\le 0.$$

It is easy to see that

$$|u_i(t)|=|\psi_i(t)|\le\|\psi\|_\infty\le Me_{\ominus\lambda}(t,t_0)\|\psi\|_\infty,\ \ t\in[-v,0]_{\mathbb{T}},\ \ i=1,2,\dots,n,$$

where $\lambda\in\mathcal{R}^+$ is the same as that in (6.2), which implies that

$$\|x(t)-x^*(t)\|=\max_{1\le i\le n}\big\{|u_i(t)|\big\}\le Me_{\ominus\lambda}(t,t_0)\|\psi\|_\infty,\ \ t\in[-v,0]_{\mathbb{T}}.$$

Next, we claim that

$$\|x(t)-x^*(t)\|\le Me_{\ominus\lambda}(t,t_0)\|\psi\|_\infty,\ \ \forall t\in(0,+\infty)_{\mathbb{T}}. \tag{6.4}$$

In order to prove (6.4), we first show for any $p>1$, the following inequality holds

$$\|x(t)-x^*(t)\|<pMe_{\ominus\lambda}(t,t_0)\|\psi\|_\infty,\ \ \forall t\in(0,+\infty)_{\mathbb{T}}. \tag{6.5}$$

If (6.5) is not true, then there must be some $t_1\in(0,+\infty)_{\mathbb{T}}$, $C>1$ and some k such that

$$\|x(t_1)-x^*(t_1)\|=|x_k(t_1)-x_k^*(t_1)|=CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty \tag{6.6}$$

and

$$\|x(t)-x^*(t)\|\le CpMe_{\ominus\lambda}(t,t_0)\|\psi\|_\infty,\quad \forall t\in[-v,t_1]_{\mathbb{T}}. \tag{6.7}$$

By (6.3)-(6.7) and (H_2)-(H_5), we obtain

$$\begin{aligned}|u_i(t_1)|\le\ & e_{-c_i}(t_1,t_0)\|\psi\|_\infty+\int_{t_0}^{t_1}CpM\|\psi\|_\infty e_{-c_i}(t_1,\sigma(s))\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j e_{\ominus\lambda}(s,t_0)\\ &+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j e_{\ominus\lambda}(s-\gamma_{ij},t_0)\bigg)\Delta s\\ \le\ & CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty\bigg\{\frac{1}{CpM}e_{-c_i}(t_1,t_0)e_{\ominus\lambda}(t_0,t_1)+\int_{t_0}^{t_1}e_{-c_i}(t_1,\sigma(s))e_{\lambda}(t_1,\sigma(s)))\\ &\times\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j e_{\ominus\lambda}(s,\sigma(s))+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j e_{\ominus\lambda}(s-\gamma,\sigma(s))\bigg)\Delta s\bigg\}\end{aligned}$$

$$
\begin{aligned}
&< CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty\bigg\{\frac{1}{M}e_{-c_i\oplus\lambda}(t_1,t_0)+\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))\\
&\quad+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j\exp\big(\lambda(\gamma+\sup_{s\in\mathbb{T}}\mu(s))\big)\bigg)\int_{t_0}^{t_1}e_{-c_i\oplus\lambda}(t_1,\sigma(s))\Delta s\bigg\}\\
&\le CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty\bigg\{\frac{1}{M}e_{-c_i\oplus\lambda}(t_1,t_0)+\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))\\
&\quad+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j\exp\big(\lambda(\gamma+\sup_{s\in\mathbb{T}}\mu(s))\big)\bigg)\frac{1-e_{-c_i\oplus\lambda}(t_1,t_0)}{\underline{c_i}-\lambda}\bigg\}\\
&\le CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty\bigg\{\bigg[\frac{1}{M}-\frac{1}{\underline{c_i}-\lambda}\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))\\
&\quad+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j\exp\big(\lambda(\gamma+\sup_{s\in\mathbb{T}}\mu(s))\big)\bigg)\bigg]e_{-c_i\oplus\lambda}(t_1,t_0)\\
&\quad+\frac{1}{\underline{c_i}-\lambda}\bigg(\sum_{j=1}^{n}\overline{a_{ij}}\alpha_j\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))+\sum_{j=1}^{n}\overline{b_{ij}}\alpha_j\exp\big(\lambda(\gamma+\sup_{s\in\mathbb{T}}\mu(s))\big)\bigg)\bigg\}\\
&< CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty. \qquad (6.8)
\end{aligned}
$$

(6.8) implies that

$$|x_k(t_1)-x_k^*(t_1)| < CpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_\infty,\ \ \forall k\in\{1,2,\dots,n\},$$

which contradicts (6.6), and so (6.5) holds. Letting $p\to 1$, then (6.4) holds. Hence, the weighted pseudo-almost periodic solution of system (1.1) is globally exponentially stable. The global exponential stability implies that the weighted pseudo-almost periodic solution is unique. ∎

Corollary 6.1. *If conditions* (H_1)*-*(H_3) *and* (H_5) *hold. Furthermore, suppose that* $I_i(i=1,2,\dots,n)$ *are almost periodic functions, then system* (1.1) *has a unique almost periodic solution which is globally exponential stable.*

Corollary 6.2. *If conditions* (H_1)*-*(H_3) *and* (H_5) *hold. Furthermore, suppose that* $I_i(i=1,2,\dots,n)$ *are pseudo-almost periodic functions, then system* (1.1) *has a unique pseudo-almost periodic solution which is globally exponential stable.*

7 Numerical examples

Consider the following neural network:

$$x_i^{\Delta}(t)=-c_i(t)x_i(t)+\sum_{j=1}^{2}a_{ij}(t)f_j(x_j(t))+\sum_{j=1}^{2}b_{ij}(t)f_j(x_j(t-\gamma_{ij}))+I_i(t),\ i=1,2, \qquad (7.1)$$

where $f_1(x) = \frac{\cos^3 x+5}{18}$, $f_2(x) = \frac{\cos^3 x+3}{12}$ and the weight $u = \frac{1}{2} + e^{-|t|}$.

Example 7.1. *Take* $\mathbb{T} = \mathbb{R}$ *and*

$$c_1(t) = 11 + |\cos(\sqrt{2}t)|, \quad c_2(t) = 12 - |\sin t|,$$

$$I_1(t) = \frac{2}{16}(\sin t + \sin(t + \frac{\pi}{6})), \quad I_2(t) = \frac{\sqrt{2}}{8}(\sin t + \sin(\frac{\pi}{4} + \sqrt{2}t)),$$

$$(a_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{15}{7}|\cos t| & \frac{10}{7}|\sin t| \\ \frac{18}{7}|\cos t| & \frac{13}{14}|\sin t| \end{pmatrix}, \ (b_{ij}(t))_{2\times 2} = \begin{pmatrix} 2|\sin t| & \frac{5}{3}|\cos t| \\ \frac{10}{3}|\sin t| & \frac{1}{24}|\sin t| \end{pmatrix}.$$

Let $\gamma_{ji}(i,j = 1,2)$ *be real numbers, then* $(H_2) - (H_4)$ *hold. Let* $\alpha_1 = \alpha_2 = \frac{1}{4}$*, then* (H_1) *holds. Next, let us check* (H_5)*, if we take* $r_0 = 1$*, then*

$$\max\{\frac{\eta_1}{\underline{c_1}}, \frac{\eta_2}{\underline{c_2}}\} + L = \frac{1064}{2772} + \frac{\sqrt{2}}{44} \approx 0.566 < 1 = r_0$$

and

$$\max\{\Pi_1, \Pi_2\} = \max\{\frac{152}{84}, \frac{165}{96}\} = \frac{152}{84} < 11 = \min\{\underline{c_1}, \underline{c_2}\}.$$

Thus, (H_5) *holds for* $r_0 = 1$*. Now, by Theorem 5.1 and Theorem 6.1, system* (7.1) *has a unique weighted pseudo-almost periodic solution in the region* $E = \{\varphi \in PAP(\mathbb{T}, \mathbb{R}^2, u) : \|\varphi\|_\infty \le 1\}$*, which is globally exponential stable.*

Example 7.2. *Take* $\mathbb{T} = \mathbb{Z}$ *and*

$$c_1(t) = 0.9 - 0.1|\sin(\sqrt{3}t)|, \ c_2(t) = 0.8 + 0.1\cos^2 t,$$

$$I_1(t) = \frac{1}{32}(3\sin t + \sqrt{3}\cos t), \quad I_2(t) = \frac{1}{64}(\sqrt{2}\sin t + \sqrt{2}\cos t + 2\sin t),$$

$$(a_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{1}{7}|\sin t| & \frac{1}{7}\sin^2 t \\ \frac{3}{14}|\cos t| & \frac{1}{14}|\sin(\sqrt{2}t)| \end{pmatrix}, \ (b_{ij}(t))_{2\times 2} = \begin{pmatrix} \frac{1}{8}|\sin t| & \frac{1}{24}\cos^2 t \\ \frac{1}{48}|\sin t| & \frac{1}{16}|\cos t| \end{pmatrix}.$$

Let $\gamma_{ji}(i,j = 1,2)$ *be arbitrary nature numbers, then* (H_2)*-*(H_4) *hold. Let* $\alpha_1 = \alpha_2 = \frac{1}{4}$*, then* (H_1) *holds. Next, let us check* (H_4)*, if we take* $r_0 = 1$*, then*

$$\max\{\frac{\eta_1}{\underline{c_1}}, \frac{\eta_2}{\underline{c_2}}\} + L = \frac{665}{2016} + \frac{5}{16} \approx 0.642 < 1 = r_0$$

and

$$\max\{\Pi_1, \Pi_2\} = \max\{\frac{19}{168}, \frac{31}{336}\} = \frac{19}{168} < 0.8 = \min\{\underline{c_1}, \underline{c_2}\}.$$

Thus, (H_5) *holds for* $r_0 = 1$*. Now, by Theorem 5.1 and Theorem 6.1, system* (7.1) *has a unique weighted pseudo-almost periodic solution in the region* $E = \{\varphi \in PAP(\mathbb{T}, \mathbb{R}^2, u) : \|\varphi\|_\infty \le 1\}$*, which is globally exponential stable.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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